Poles and Zeros

Physical meaning

Poles

A system
1. In which energy is stored
2. That is not in its quiescent state
3. To which no excitations have been applied

tends to a quiescent state in which the energy is distributed over, dissipated in, and/or radiated by the system

The poles of the system are the complex eigenfrequencies that describe this process; their number is equal to the number of independent energy storage elements in the system.

Zeros

The zeros are the complex frequencies at which the transfer, from a specific input to a specific output of the system, equals zero.

Mathematical description

Lumped, stationary, linear, dynamic systems

Linear
1. Property of homogeneity
2. Property of additivity (superposition)

Dynamic
1. Responses at some time instant depend on excitations at the same time instant and on previous (causal) or future (non-causal, predictive) values of excitations

Linear differential equations

Single input and single output system

The sum of a number of derivatives of the response equals the sum of a number of derivatives of the excitation.
- excitation x(t)
- response y(t)
- lumped: ordinary differential equations
- stationary: fixed (time-independent) coefficients

$$\sum_{i=0}^{i=n} a_i \frac{d^i y(t)}{dt^i} = \sum_{j=0}^{j=m} b_j \frac{d^j x(t)}{dt^j}$$

Poles
The solution of the homogeneous differential equation:

$$\sum_{i=0}^{i=n} a_i \frac{d^i y(t)}{dt^i} = 0$$

Since exponential functions retain their shape under integration and differentiation this solution (the response) consists of a number of exponential functions.

Laplace Transform

Laplace: any signal x(t) can be written as an infinite sum of complex exponentials, each element of which has its own complex amplitude X(s) and complex frequency s.

$$x(t) = \frac{1}{2\pi j} \oint_{\sigma - j\omega}^{\sigma + j\omega} X(s) \exp(st) ds$$

Transfer function

By using this signal modeling technique, the differentail equation converts into an algebraic equation. A transfer function can be defined that relates the complex amplitudes of the complex exponentials of the response to those of the excitation:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{j=0}^{j=m} b_j s^j}{\sum_{i=0}^{i=n} a_i s^i}$$

Poles
Solutions for s of the denominator of H(s)
Zeros
Solutions for s of the numerator of H(s)

Characteristic equation (polynomial)

The poles are the solutions of the characteristic equation:

$$\sum_{i=0}^{i=n} a_i s^i = 0$$

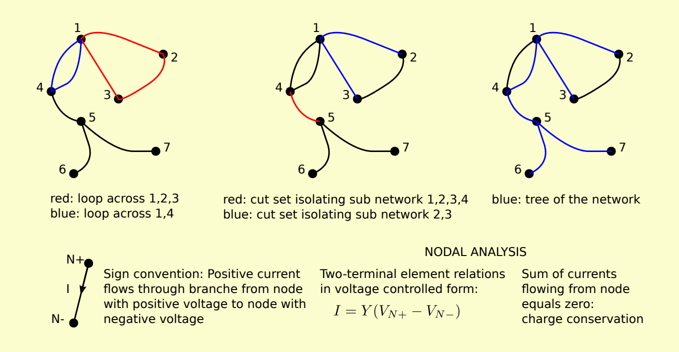
Companion matrix
The companion matrix can be used to describe the characteristic equation as a set of first-order equations:

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix}$$

$$\det(sI - A) = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0$$

Eigenvalues
The eigenvalues of the companion matrix are the roots of the characteristic polynomial.

Network theory



Graph theory

1. A graph consists of branches and nodes
2. A connected graph has at least one path among the branches that connects all the nodes
3. A sub-graph is a subset of branches with their corresponding nodes
4. A closed path of branches is called a loop
5. A collection of branches that isolates a sub-graph when removed, is called a cut set
6. A tree is a collection of branches that connects all the nodes but has no loops

Incidence matrix A
Topology information about branches and nodes; rows: nodes, columns: branches
Element matrix B
Matrix with element node-branch relations
Column vector with independent variables V
Nodal voltages or branch currents
Column vector with dependent variables I
Nodal currents and branch voltages (independent sources)
Network matrix equation
- passive network:

$$\mathbf{I} = \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{A}^T \cdot \mathbf{V}$$

Nodal analysis

Voltage-controlled elements only
Admittance matrix Y:

$$\mathbf{I} = \mathbf{Y} \cdot \mathbf{V}$$

General form of the admittance matrix Y:

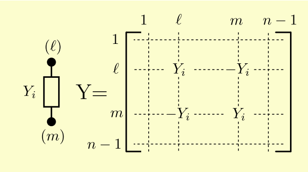
$$\sum i_k = -\sum \mathbf{Y}_{k,1} v_1 - \sum \mathbf{Y}_{k,2} v_2 \quad \dots + \sum \mathbf{Y}_{k,k} v_k \quad \dots - \sum \mathbf{Y}_{k,n-1} v_{n-1}$$

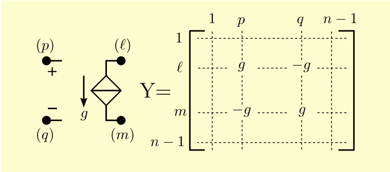
$\sum i_k$ Sum of independent currents flowing into node k

$\sum \mathbf{Y}_{k,j}$ Sum of admittances connected between node k and j

$\sum \mathbf{Y}_{k,k}$ Sum of admittances connected to node k

v_k Voltage at node k

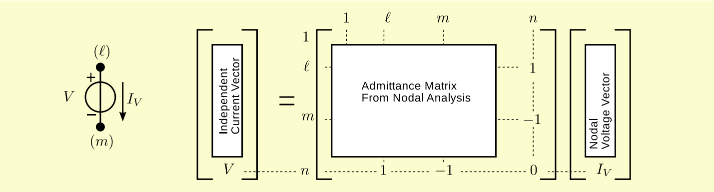




Modified nodal analysis

Voltage-controlled and current-controlled elements
- Add a row with the branch relation
- Add a column with the unknown current

$$\mathbf{I} = \mathbf{M} \cdot \mathbf{V}$$



Transfer function

Transfer from independent variable to dependent variable

$$\frac{\mathbf{V}_j}{\mathbf{I}_k} = \mathbf{M}_{j,k}^{-1} = \frac{(-1)^{j+k} \det(\mathcal{M}_{k,j})}{\det(\mathbf{M})}$$

Minor matrix: $\mathcal{M}_{k,j}$ equals M after leaving out row k and column j.

Poles
Solutions for s of $\det(\mathbf{M}) = 0$
Zeros
Solutions for s of $\det(\mathcal{M}_{k,j}) = 0$

The time-constant matrix

Basis for intuitive determination of poles in systems without feedback
General form of the MNA matrix, written as first-order equations of s

$$\mathbf{M} = \mathbf{G} + s\mathbf{C}$$

Characteristic equation: $\det(\mathbf{G} + s\mathbf{C}) = 0$

Generalized eigenvalue problem

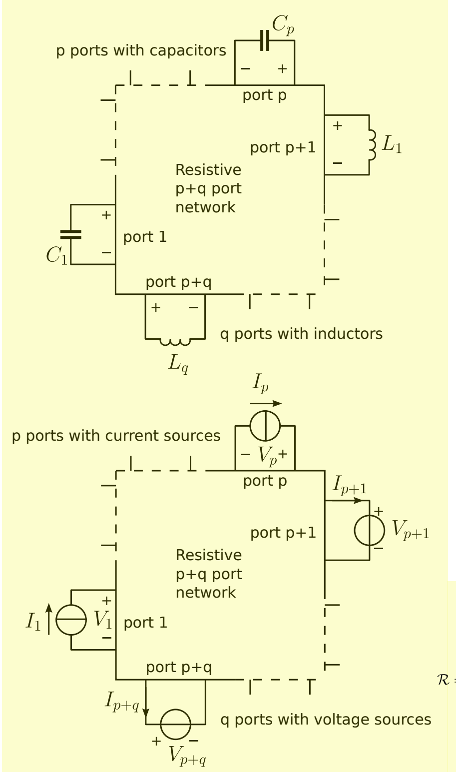
$$\det(\mathbf{I} + \lambda \mathbf{T}) = 0 \qquad \mathbf{T} = \mathbf{G}^{-1} \cdot \mathbf{C}$$

Poles and eigenvalues of the time-constant matrix:

If τ_i is an eigenvalue of $\mathbf{T}$ then $p_i = -\frac{1}{\tau_i}$ is a pole of the network.

Number of poles and the dimension of T
- The number of poles equals the number of independent state variables, which is the sum of the number of independent capacitor voltages and the independent inductor currents
- The dimension of T as defined above, is equal to the dimension of M, which is the number of nodes minus one; this is usually much larger than the number of independent state variables.
- Since the number of eigenvalues of a (square) matrix equals its number of rows or columns we may find a lot of poles at infinity, and reduction of the dimension of the time-constant matrix to the number of state variables is desired (numerical stability).
- Unfortunately there is no straightforward method to do this, but we can reduce its dimension of the time-constant matrix to the sum of the number of inductors and capacitors.

Structure and meaning of the time-constant matrix



Multi-port resistive network
- Capacitors and inductors connected the ports of a multi-port resistive network
- Energy stored on the element connected to a port is dissipated in the resistive network or interacts with energy stored in elements connected to other ports
- Such interactions complicate the estimation of the poles
- If there are no interactions only the diagonal elements of the time-constant matrix have a non-zero value
- The trace of the matrix then equals the sum of its eigenvalues
- Poles are found from the non-zero values of the diagonal elements
- The time-constant matrix can only be used for networks that have a DC solution

$$\mathcal{R} = \begin{pmatrix} r_{11} & \cdots & r_{1p} & \mu_{11} & \cdots & \mu_{1q} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ r_{p1} & \cdots & r_{pp} & \mu_{p1} & \cdots & \mu_{pq} \\ \alpha_{11} & \cdots & \alpha_{1p} & g_{11} & \cdots & g_{q1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{q1} & \cdots & \alpha_{qp} & g_{1q} & \cdots & g_{qq} \end{pmatrix}$$

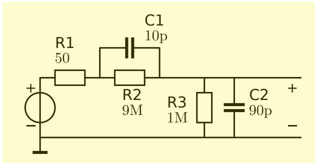
Procedure for finding poles from the time-constant matrix

Accurate: calculate eigenvalues
1. Be sure the network has a DC solution:
   - No cut sets of capacitors and/or current sources:
     If not, insert resistors with a large resistance in parallel with the branches of such independent cut sets
   - No loops of inductors and/or voltage sources:
     If not, insert resistors with a low resistance in series with the branches of such independent loops
2. Determine the resistance matrix through network inspection
3. Determine the time-constant matrix by multiplying the resistance matrix with the diagonal matrix with reactive port elements
4. Calculate the eigenvalues of the time-constant matrix

Estimation: ignore interaction
1. Be sure the network has a DC solution:
   - No cut sets of capacitors and/or current sources:
     If not, insert resistors with a large resistance in parallel with the branches of such independent cut sets
   - No loops of inductors and/or voltage sources:
     If not, insert resistors with a low resistance in series with the branches of such independent loops
2. Determine the resistance of each port of the multi-port resistive network
3. For each port determine the time constant that follows from its port resistance and its reactive element
4. The largest time-constant is the one of the dominant pole, denote it and:
   a. Short the port of this largest time constant in cases in which a capacitor was connected to it, or leave it open in cases in which an inductor was connected to it.
   b. Determine the port resistances of the modified network, find the next dominant time constant, etc. ...

Note:
If a dominant time constant is found at a port at which a capacitor is connected, then for frequencies above the frequency associated with the dominant time constant, the capacitor can be considered to short the port.
If a dominant time constant is found at a port at which an inductor is connected, then for frequencies above the frequency associated with the dominant time constant, the inductor can be considered to be disconnected from the port.

Network probe circuit



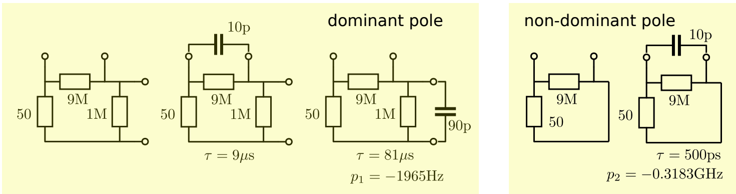
Time-constant matrix and eigenvalues

$$\begin{pmatrix} \frac{9\cdot10^6(50+10^6)}{50+10\cdot10^6} & -\frac{9\cdot10^6\times10^6}{50+10\cdot10^6} \\ -\frac{9\cdot10^6\times10^6}{50+10\cdot10^6} & \frac{10^6(50+9\cdot10^6)}{50+10\cdot10^6} \end{pmatrix} \begin{pmatrix} 10^{-11} & 0 \\ 0 & 90\cdot10^{-12} \end{pmatrix}$$

R-matrix C-matrix

$p_1 = -1768\text{Hz}$ $\quad p_2 = -0.3537\text{GHz}$

Multi-port resistive networks and port time constants

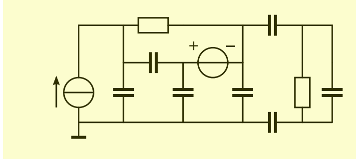


Ignored interaction: product of the poles is correct, sum of the poles too small, interaction increases the absolute value of their sum.

Estimation of poles and zeros in networks without feedback

Number of poles
- The number of poles equals the sum of the number of independent capacitor voltages and independent inductor currents
  - The number of independent capacitor voltages equals the number of capacitors minus the number of independent loops of capacitors or capacitors and voltage sources
    - The number of independent loops of capacitors or capacitors and voltage sources equals the number of capacitors and/or voltage sources that must be removed from the network to break all the loops
  - The number of independent inductor currents equals the number of inductors minus the number of independent cut sets of inductors and inductors and current sources
    - The number of independent cut sets of inductors and inductors and current sources equals the number of inductors and/or current sources that need to be replaced in the network to obtain a connected graph, after all inductors and current sources that are part of such a cut set have been removed

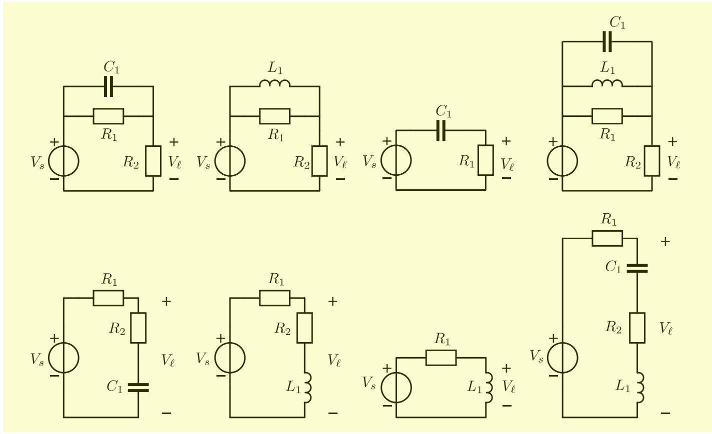
Number of poles at zero frequency
- The number of poles at zero frequency equals the number of independent cut sets of capacitors and capacitors and current sources plus the number of independent loops of inductors and inductors and voltage sources

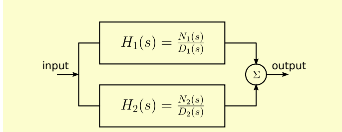


Network with 7 capacitors
1. How many independent loops of capacitors and voltage sources?
2. How many poles?
3. How many independent cut sets of capacitors of or capacitors and current sources?
4. How many poles at s=0?

Procedure for finding zeros

Zero transfer at complex frequency
- Short in parallel with the signal path
- Open circuit in series with the signal path
- Multiple transfer paths that cancel each other at some complex frequency





$$H_1(s) + H_2(s) = \frac{N_1(s)D_2(s) + N_2(s)D_1(s)}{D_1(s)D_2(s)}$$

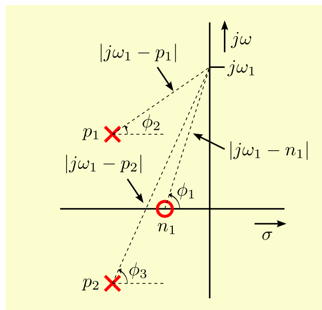
Zero transfer at zero frequency
- DC short in parallel with the signal path
- DC open circuit in series with the signal path
- Multiple transfer paths that cancel each other at DC

Relation between poles and zeros and Bode plots

The y-axis in the complex plane corresponds with the frequency axis in the Bode plots.

$$|H(j\omega)| = \frac{b_m}{a_n} \frac{\prod_{i=0}^{i=m} |j\omega - z_j|}{\prod_{k=0}^{k=n} |j\omega - p_i|}$$

$$\arg\{H(j\omega)\} = \arg b_m - \arg a_n + \sum_{i=0}^{j=m} \arg(j\omega - z_j) - \sum_{i=0}^{i=n} \arg(j\omega - p_i)$$



Relation between poles and zeros and the unit-impulse response

The poles of the system are the coefficients of t in the exponentials of the unit-impulse response.

A stable system has all poles in the negative half of the complex plane.

$$h(t) = \sum_{i=1}^{n} \sum_{j=0}^{\ell-1} A_{i,j} t^j \exp p_i t$$