Selected Topics from System modeling

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Systems

Most physical systems show nonlinear, dynamic, time-variant and distributed behavior.
Their modeling, however, is to the convenience of the modeler.
G.P. Box: 'All models are wrong, but some are useful'.

$$
\begin{aligned}
linear &\Longleftrightarrow nonlinear \\
instantaneous &\Longleftrightarrow dynamic \\
fixed &\Longleftrightarrow time - variant \\
causal &\Longleftrightarrow non - causal \\
stable &\Longleftrightarrow instable \\
lumped &\Longleftrightarrow distributed
\end{aligned}
$$

- Intended behavior of many information processing systems: linear, fixed, instantaneous, causal and stable
- Dynamic effects can be investigated by considering it linear
- Nonlinearity can be investigated by considering it instantaneous
- Nonlinear dynamic behavior can be modelled for special cases
- In this course only lumped systems

Systems

Linear, instantaneous, fixed systems

- Input signal (excitation): $x(t)$, output signal (response) $y(t)$.
- Relation between response and excitation can be described with one constant G: system gain

$$y(t) = G \cdot x(t)$$

Systems

Linear, dynamic, fixed systems

- Description with normal linear differential equations

$$\sum_{k=0}^{k=n} a_k \frac{d^k y(t)}{dt^k} = \sum_{i=0}^{i=m} b_i \frac{d^i x(t)}{dt^i}$$

- Coefficients are constants
- Solution with the aid of resolved signals
  - Time-domain: resolution in unit-impulse or unit-step functions, convolution
  - Frequency-domain modeling: resolution in imaginary exponentials: Cosine transform, Fourier series, Fourier Transform
  - Complex frequency-domain modeling: resolution into elementary complex exponentials: Laplace transform.

Linear, dynamic, fixed system analysis

Modeling and analysis with time-domain resolved signals

Modeling:

- Unit impulse response: $\mathcal{H}\left\{\delta(t)\right\} = h(t)$
- Unit-step response: $\mathcal{H}\left\{\mu(t)\right\} = a(t)$

Analysis:

Convolution:

$$y(t) = h(t) * x(t) = \int_0^t h(t-\tau)x(\tau)d\tau$$

$$y(t) = a(t) * \dot{x}(t) = \int_0^t a(t-\tau)\dot{x}(\tau)d\tau$$

Measurement:

Response to a periodic pulse can easily be related to unit-step response.

Linear, dynamic, fixed systems

Modeling and analysis with frequency-domain resolved signals

- Uses resolution of pulse signals into imaginary exponentials
- Exponentials retain their shape under differentiation
- Differential equation is transformed into a power series in $j\omega$.
- Transfer function $H(j\omega)$ relates the complex amplitude of component of the output signal to component of the excitation:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}, \; H(j\omega) = \mathcal{F}\left\{h(t)\right\}$$

$$\frac{S_y(\omega)}{S_x(\omega)} = \left|H(j\omega)\right|^2$$

Linear, dynamic, fixed systems

Modeling and analysis with complex frequency-domain resolved signals

- Uses resolution of signals into complex exponentials
- Exponentials retain their shape under differentiation
- Differential equation is transformed into a power series in s.
- System function $H(s)$ relates the complex amplitude of one component of the output signal to the corresponding component of the excitation:

$$Y(s) = H(s)X(s), \; H(s) = \mathcal{L}\left\{h(t)\right\}$$

- Time domain analysis using Laplace:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{H(s)X(s)\} = \mathcal{L}^{-1}[H(s)\mathcal{L}\{x(t)\}]$$

Linear, dynamic, fixed systems

Fourier versus Laplace

- Both transformation methods can be used for solving linear time invariant differential equations
- Fourier: all signals must be absolute integrable:

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

  - Cannot be used for instable systems with theoretically unbounded response to bounded input

Linear, dynamic, fixed systems

Characterization of the system function

System function $H(s)$; a_k and b_i coefficients of differential equation:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{i=0}^{i=m} b_i s^i}{\sum_{k=0}^{k=n} a_k s^k}$$

Characterization with poles and zeros:

$$H(s) = \frac{b_0}{a_0} \frac{\prod_{i=1}^{m} (1 - s/z_i)}{\prod_{k=1}^{n} (1 - s/p_k)} \text{ or } H(s) = \frac{b_m}{a_n} \frac{\prod_{i=0}^{m-1} (s - z_i)}{\prod_{k=0}^{n-1} (s - p_k)}$$

Zero frequency transfer $H(0)$ if $a_0 \neq 0,:$ $H(0) = \frac{b_0}{a_0}$

Transfer for $s \to \infty$, $(n > m):$ $H(s)_{s \to \infty} = \frac{b_m}{a_n s^{n-m}}$

Linear, dynamic, fixed systems

Impulse response

1. System has k poles, of which ℓ different (non coinciding)
2. A pole p_ℓ occurs q times

$$h(t) = \sum_{i=1}^{\ell} \left(\sum_{j=1}^{q} \frac{A_j}{(j-1)!} t^{j-1} \exp p_\ell t \right)$$

3. A_j follows from partial fraction expansion (residues)

Stable if $\mathrm{Re}(p_\ell) < 0$ for all ℓ; poles in left half of $s - plane$.

Relation between pole-zero diagram and step response

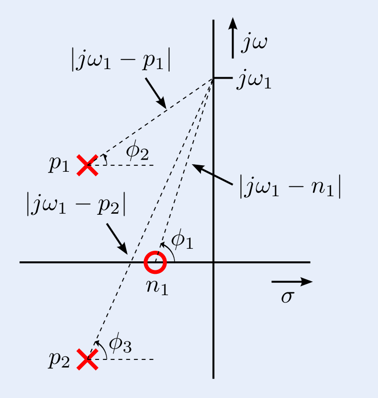
- Laplace transform of unit step: $\mathcal{L}\{\mu(t)\} = \frac{1}{s}$
- Laplace transform of the unit impulse is the system function $H(s)$
- Unit step response : $a(t)$
- Initial value theorem: $a(0^+) = \lim_{s\to\infty} s\frac{1}{s}H(s) = \lim_{s\to\infty} H(s)$
- Final value theorem: $a(\infty) = \lim_{s\to 0} s\frac{1}{s}H(s) = \lim_{s\to 0} H(s)$

Linear, dynamic, fixed systems

Relations between description methods

Relation between pole-zero diagram and Bode Plots: $s = j\omega$.

$$H(j\omega) = \frac{b_m}{a_n} \frac{\prod_{i=0}^{m-1}(j\omega - z_i)}{\prod_{k=0}^{n-1}(j\omega - p_k)}$$



System modeling

Nonlinear, instantaneous, fixed systems

- Modeling with the aid of Taylor series
- Instantaneous behavior = frequency-independent behavior
- Characterization: various description methods for deviation from linear behavior exist.
- Description methods that closely relate to the observer's error perception, should be used.

- Offset
- Gain
- Nonlinearity
- Differential-gain
- Harmonic distortion
- Intermodulation distortion
- Gain compression

Nonlinear instantaneous fixed

Description with Taylor series

Taylor expansion in origin of a static nonlinear function $y(x)$:

$$y(x) = a_o + a_1 x + a_2 x^2 + \ldots a_n x^n$$

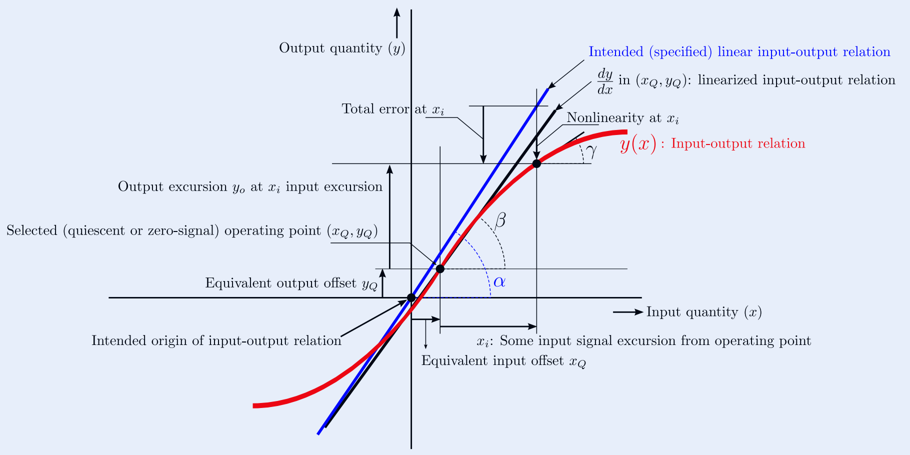
$$a_n = \frac{1}{n!}\frac{d^n y}{dx^n}$$

a_0 : offset

a_1 : gain

Nonlinear instantaneous fixed

Static input-output relation



Nonlinear instantaneous fixed

Harmonic and intermodulation distortion

- Higher order terms from Taylor series cause offset, harmonic distortion and intermodulation distortion:

$$\cos^2 \omega t = \frac{1}{2} + \frac{1}{2} \cos 2\omega t$$

$$\begin{aligned}(\cos \omega_1 t + \cos \omega_2 t)^2 &= \cos^2 \omega_1 t + \cos^2 \omega_2 t + \\ &+ \cos(\omega_1 - \omega_2) t + \cos(\omega_1 + \omega_2) t\end{aligned}$$

- Total harmonic distortion (THD): ratio of the power of all harmonics and of the fundamental
- Intermodulation distortion (IMD), two fundamentals with equal amplitude: ratio of the power of the intermodulation component and that of one fundamental. Second order IMD: $\omega_1 \pm \omega_2$, third-order IMD $2\omega_1 \pm \omega_2$ or $\omega_1 \pm 2\omega_2$ etc.

Nonlinear instantaneous fixed

Harmonic and intermodulation distortion and differential gain

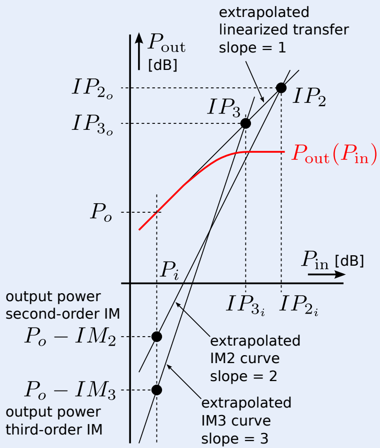
For instantaneous fixed nonlinear systems there exists a relation between the harmonic and intermodulation distortion and the differential gain:

$$d_2 = \frac{\varepsilon^+ - \varepsilon^-}{8},\ d_3 = \frac{\varepsilon^+ + \varepsilon^-}{24}$$
$$IM_2 = \frac{\varepsilon^+ - \varepsilon^-}{4},\ IM_3 = \frac{\varepsilon^+ + \varepsilon^-}{8}$$

ε^+ and ε^- are he differential gain at positive and negative peak excursion (considering one carrier for IMD), respectively.

Nonlinear instantaneous fixed

Intermodulation intercept points



System modeling

Nonlinear, dynamic, fixed systems

1. Differential-gain: depends on frequency
2. Differential phase: depend on frequency
3. THD: depend on frequency; relations between the differential gain and the second and third order harmonic distortion are not longer valid.
4. Intermodulation: depends on frequency; relations between the differential gain and the second and third order intermodulation distortion are not longer valid.
5. Gain compression: depends on frequency; relation between the IP3 and the 1 [dB] compression point is not longer valid.
6. Slew rate limitation:

$$SR^{+} = \left.\frac{dy(t)}{dt}\right|_{\text{max}}; \text{ if } \frac{dy(t)}{dt} \geq 0$$
$$SR^{-} = -\left|\frac{dy(t)}{dt}\right|_{\text{max}}; \text{ if } \frac{dy(t)}{dt} < 0$$

Slew rate and full-power bandwidth

Limiting and dynamic effects

- Limiting voltage with inductive load causes a limited rate of change of the current:

$$\left.\frac{di}{dt}\right|_{\max} = \frac{V_{\max}}{L}$$

- Limiting current with capacitive load causes a limited rate of change of the current:

$$\left.\frac{dv}{dt}\right|_{\max} = \frac{I_{\max}}{C}$$

- Full-power bandwidth: maximum frequency at which no slew-rate distortion can be observed, at a given amplitude of a sinusoidal output signal.