Selected Topics from Signal Modeling

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Signal modeling

The signal and its model

- A signal is a physical quantity that contains meaningful data.
- Signal models are required for the specification, design and verification of signal processing systems.
- One may decide to use either random or deterministic signal models.
  - Deterministic signal modeling techniques use known time functions.
  - Random signal modeling techniques use statistical description methods:
    - Probability density function
    - Average, expectation
    - Mean square value, variance
    - Root Mean Square value, standard deviation
    - Power spectral density
- Signals are signals; they are not random nor deterministic. There models are!

Deterministic signal modeling

Classification of signals: power and energy signals

- A signal with a nonzero average power is called a *power signal*.
- The average power of a signal $x(t)$ is proportional to the mean square value $\overline{x(t)^2}$ of that signal:

$$\overline{x(t)^2} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \{x(t)\}^2 dt$$

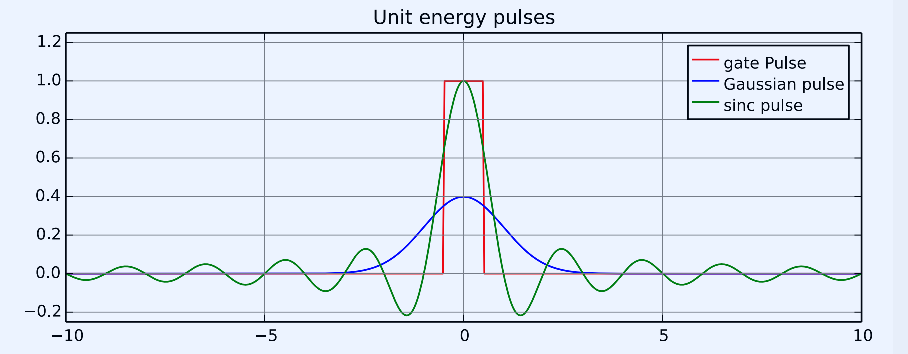
- If the mean square value of a signal equals zero, it is called an *energy signal*. The energy $W\{x(t)\}$ of a signal $x(t)$ is defined as:

$$W\{x(t)\} = \lim_{T\to\infty} \int_{-T/2}^{T/2} \{x(t)\}^2 dt$$

- In practice we observe the signal over a limited observation time: $0 \leq t \leq T$.

Examples of unit-energy pulse signals

Pulse signals: signal energy is concentrated in limited time interval.



Examples of unit-energy pulse signals

Unit gate pulse signal $G(t,\tau)$:

$$G(t,\tau) = \frac{1}{\tau}, |t| < \frac{\tau}{2}$$
$$G(t,\tau) = 0, \ \ |t| \geq \frac{\tau}{2}$$

Unit Gaussian pulse signal $x(t,\tau)$:

$$x(t,\tau) = \frac{1}{\tau\sqrt{2\pi}} \exp\left(\frac{-t^2}{2\tau^2}\right)$$

Unit sinc pulse signal $\mathrm{sinc}(t,\tau)$:

$$\mathrm{sinc}(t,\tau) = \frac{\sin(\pi t/\tau)}{\pi t}$$

Unit impulse function: $\lim_{\tau\to 0}$ of all three above pulse signals:

$$\int_{-\infty}^{\infty} \delta(t) = 1, \ \delta(t) = 0 \text{ for } t \neq 0$$

Resolution of signals into elementary signals

- Many information processing systems are intended to be linear:

  If an arbitrary signal is resolved into basic signals the response to this signal can be written as the sum of the responses to the components.

- Time domain analysis: resolution in unit impulse or unit step functions
- Exponential functions retain their shape under differentiation and integration. Resolution in exponential functions eases the analysis of linear dynamic systems:
  - Frequency domain analysis: resolution in imaginary exponentials (Fourier)
  - Complex frequency domain analysis: resolution in imaginary exponentials (Laplace)

Time domain signal modeling

Resolution of signals in unit impulse or in unit step functions

- A signal $x(t)$ can be resolved into a continuum of delayed unit impulses: Assume $x(t)$ only has nonzero values between $t = 0$ and $t = T$:

$$x(t) = \int_0^T x(\tau)\delta(t-\tau)d\tau$$

- The resolution of a signal in unit step functions $\mu(t)$ can be found in a similar way:

$$x(t) = \int_0^T \dot{x}(\tau)\mu(t-\tau)d\tau$$

in which $\dot{x}(t) = \frac{d}{dt}\{x(t)\}$.

Fourier Series

Resolution of periodic signals into a series of imaginary exponentials

- Fourier series of a periodic signal:

$$x(t) = \sum_{n=-\infty}^{n=\infty} X_n \exp(jn\omega_0 t)$$

- The complex amplitude X_n of the n-th element of the series, is found as:

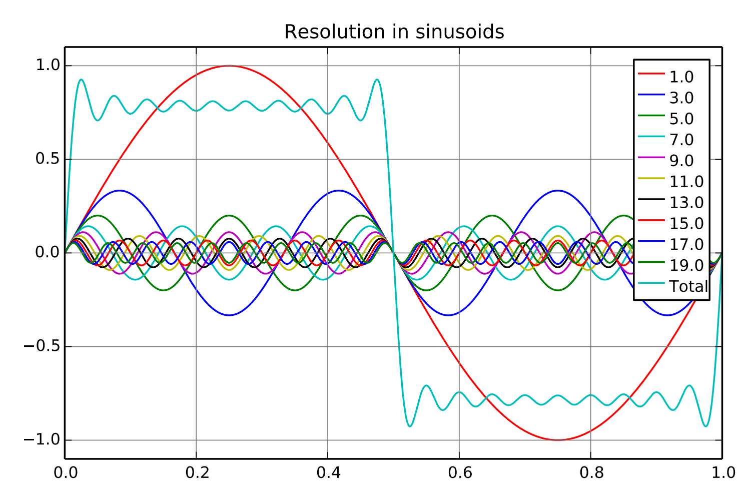
$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)dt$$

- Mean square value can be obtained from frequency domain description:

$$\overline{\{x(t)\}^2} = \sum_{n=-\infty}^{n=\infty} |X_n|^2$$

Frequency domain modeling of periodic signals

A square wave contructed from 10 sinusoidal signals



Cosine transform

Resolution of periodic signals in sinusoidal signals

- Two complex conjugated imaginary exponentials constitute a real sinusoid:

$$x(t) = x_{DC} + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t - \varphi_n)$$

- $C_n = \sqrt{A_n^2 + B_n^2}$, and $\varphi_n = \arctan \frac{B_n}{A_n}$ with:

$$A_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos(n\omega_0 t)\ dt$$

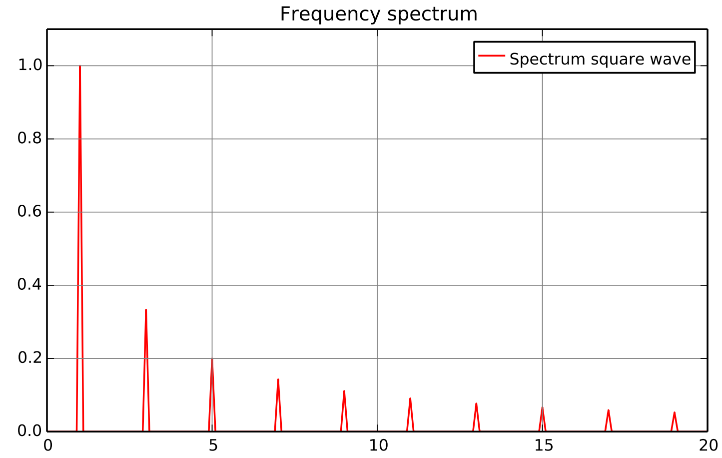
$$B_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin(n\omega_0 t)\ dt$$

- Mean square value can be obtained from frequency domain description:

$$\overline{\{x(t)\}^2} = x_{DC}^2 + \frac{1}{2} \sum_{n=1}^{n=\infty} C_n^2$$

Frequency spectrum of periodic signals

Frequency spectrum of the square wave



Fourier Transform

Resolution of pulse signals in a continuum of imaginary exponentials

- Time domain description is known as the inverse Fourier transform:

$$x(t) = \mathcal{F}^{-1}\left\{X(j\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\exp(j\omega t)d\omega$$

- The complex amplitudes $X(j\omega)$ of the exponentials is the Fourier transform $\mathcal{F}\left\{x(t)\right\}$ of $x(t)$:

$$X(j\omega) = \mathcal{F}\left\{x(t)\right\} = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)dt$$

- The pulse energy can be obtained from $X(j\omega)$:

$$W\left\{x(t)\right\} = \lim_{T\to\infty}\int_{-T/2}^{T/2}\left\{x(t)\right\}^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|X(j\omega)\right|^2 d\omega$$

Laplace Transform

Resolution of time signals in complex exponentials

- Laplace: a time-signal can (for $t \geq 0$) be represented by an infinite sum of complex exponentials $X(s)\exp(st)$:

$$x(t) = \mathcal{L}^{-1}\left\{X(s)\right\} = \frac{1}{2\pi j}\oint_{\sigma-j\omega}^{\sigma+j\omega} X(s)\exp(st)ds$$

- The complex amplitudes $X(s)$ of these complex exponentials is the Laplace transform of $x(t)$:

$$X(s) = \mathcal{L}\left\{x(t)\right\} = \int_{-\infty}^{\infty} x(t)\exp(-st)dt$$

Random signal modeling

Stationary and ergodic processes

- A random variable $\underline{x}$ comprises an ensemble of samples, taken at some time instant t, from different realizations or *sample functions* of a process
- If a process is stationary, the statistical properties of the random variable do not depend on time
- If the process is ergodic, these samples may as well have been taken from one (truncated) sample function at different time instants:
    - The samples from one truncated sample function over the time interval T, are representative for an ensemble of samples taken at one time instant
    - Generation of thermal noise in resistors
    - Ergodic processes are stationary

Random signal modeling

The use of averages

Deterministic description
Time average over a time interval T of a signal $x(t)$:

$$\overline{x(t)} = \frac{1}{T}\int_0^T x(t)dt$$

Mean square value over interval T of a signal $x(t)$:

$$\overline{x(t)^2} = \frac{1}{T}\int_0^T \{x(t)\}^2 dt$$

Statistical description
Ensemble average or expectation of a random variable $\underline{x}$:

$$E\{\underline{x}(t)\} = \int_{-\infty}^{\infty} xP(\underline{x}, t)dx$$

Mean square value of a random variable $\underline{x}$:

$$E\{\underline{x}(t)^2\} = \int_{-\infty}^{\infty} x^2 P(\underline{x}, t)dx$$

Signal modeling

RMS value

Deterministic description
Root mean square (RMS) value of a signal $x(t)$ over a time interval T:

$$x_{RMS} = \sqrt{\overline{x(t)^2}} = \sqrt{\frac{1}{T}\int_0^T \{x(t)\}^2\,dt}$$

Statistical description
Root mean square (RMS) value of a random variable $\underline{x}$:

$$\underline{x}_{RMS} = \sqrt{E\{\underline{x}(t)^2\}} = \sqrt{\int_{-\infty}^{\infty} x^2 P(\underline{x}, t)\,dx}$$

Signal modeling

Correlation and auto-correlation

- Correlation tells us about the amount of similarity between random variables, or the *joined power* of random variables:

$$E\{\underline{x}(t_1)\underline{y}(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyP(x, t_1; y, t_2)dxdy$$

$$r_{xy}(\tau) = \frac{1}{T}\int_0^T x_T(t)y_T(t+\tau)dt$$

- Auto-correlation: tells us about correspondence between values at two time instants of one random variable:

$$E\{\underline{x}(t_1)\underline{x}(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2P(x, t_1; x, t_2)dx^2$$

$$r_x(\tau) = \frac{1}{T}\int_0^T x_T(t)x_T(t+\tau)dt$$

$$r_x(0) = \overline{\{x(t)\}^2}$$

Signal modeling

Wiener-Khintchine theorem, Parseval's theorem

- If a signal changes rapidly, the joined power of the signal and a delayed version of it will approach zero for a very small time delay.
- If a signal changes slowly, the joined power of the signal and a delayed version of it remains large up to a much larger time delay.
- Wiener-Khintchine: for stationary processes the power spectral density is the Fourier transform of the autocorrelation function:

$$S_x(\omega) = \mathcal{F}\left\{r_x(\tau)\right\}$$
$$r_x(\tau) = \mathcal{F}^{-1}\left\{S_x(\omega)\right\}$$

- Parseval: the mean square value of a signal can be obtained from its frequency-domain description:

$$\overline{\{x(t)\}^2} = \frac{1}{T}\int_0^T \{x(t)\}^2 dt = \int_{f_1}^{f_2} S(f)df$$