Network Theory

Two ports and pole-zero analysis

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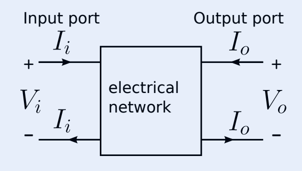
Two-ports

Introduction

In many engineering situations it is convenient to consider a network as a two-port.



- Two out of the four port variables V_i, I_i, V_o and I_o can be selected as independent variables.
- This yields six different representation methods.
- The most convenient representation can be selected for analysis.

Two-ports

Six two-port representations

$$\begin{pmatrix} V_i \\ V_o \end{pmatrix} = \mathbf{Z} \begin{pmatrix} I_i \\ I_o \end{pmatrix} \qquad \begin{pmatrix} V_i \\ I_o \end{pmatrix} = \mathbf{H}_2 \begin{pmatrix} I_i \\ V_o \end{pmatrix}$$

$$\begin{pmatrix} I_i \\ I_o \end{pmatrix} = \mathbf{Y} \begin{pmatrix} V_i \\ V_o \end{pmatrix} \qquad \begin{pmatrix} V_i \\ I_i \end{pmatrix} = \mathbf{T}_1 \begin{pmatrix} V_o \\ -I_o \end{pmatrix}$$

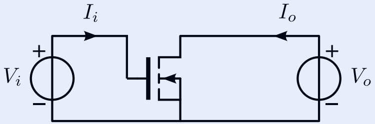
$$\begin{pmatrix} I_i \\ V_o \end{pmatrix} = \mathbf{H}_1 \begin{pmatrix} V_i \\ I_o \end{pmatrix} \qquad \begin{pmatrix} V_o \\ I_o \end{pmatrix} = \mathbf{T}_2 \begin{pmatrix} V_i \\ -I_i \end{pmatrix}$$

- Parallel connected two ports: use $\mathbf{Y}$ representation.
- Series connected two-ports use $\mathbf{Z}$ repesentation
- Amplifier design: we will use $\mathbf{T}_1$ anti-causal notation and inverse sign of I_o :

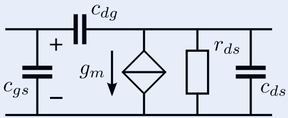
$$\begin{pmatrix} V_i \\ I_i \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V_o \\ I_o \end{pmatrix}$$

Two-ports

Three terminal example



Biased NMOS Y parameter measurement setup



Small-signal equivalent circuit

$$\begin{pmatrix} I_i \\ I_o \end{pmatrix} = \begin{pmatrix} s\left(c_{gs}+c_{dg}\right) & -sc_{dg} \\ g_m - sc_{dg} & \frac{1}{r_o} + s\left(c_{ds}+c_{dg}\right) \end{pmatrix} \begin{pmatrix} V_i \\ V_o \end{pmatrix}$$

input admittance @ storted output: $Y_{11} = \left.\frac{I_i}{V_i}\right|_{V_o=0} = s\left(c_{gs}+c_{dg}\right)$

reverse admittance @ shorted input: $Y_{12} = \left.\frac{I_i}{V_o}\right|_{V_i=0} = -sc_{dg}$

forward admittance @ shorted output: $Y_{21} = \left.\frac{I_o}{V_i}\right|_{V_o=0} = g_m - sc_{dg}$

output admittance @ shorted input: $Y_{11} = \left.\frac{I_o}{V_o}\right|_{V_i=0} = \frac{1}{r_o} + s\left(c_{ds}+c_{dg}\right)$

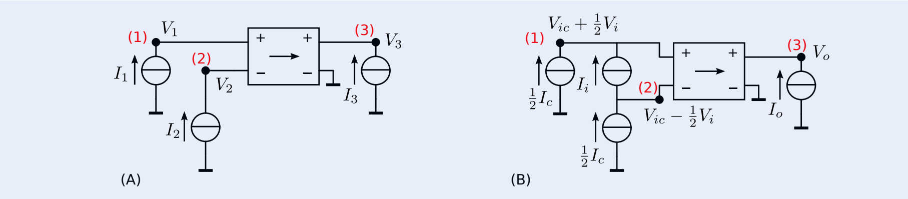
Two-ports

Validity of two-port description

- A four terminal network requires a 3×3 matrix description.
- Under certain conditions a 2×2 two-port matrix description suffices.
- This is always the case if
  1. Both ports are terminated with isolated one-ports
  2. The two-port is a natural two-port:
     - Ideal transformer
     - Gyrator
     - Ideal controlled sources.

Two-ports

Two-port constraints



$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$\begin{bmatrix} I_i \\ I_c \\ I_o \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\left(Y_{11}-Y_{12}-Y_{21}+Y_{22}\right) & 0 & \frac{1}{2}\left(Y_{13}-Y_{23}\right) \\ 0 & 0 & 0 \\ \frac{1}{2}\left(Y_{31}-Y_{32}\right) & 0 & Y_{33} \end{bmatrix} \begin{bmatrix} V_i \\ V_c \\ V_o \end{bmatrix}$$

Poles and zeros

Why should we be capable of estimating poles and zeros of networks

- Fast estimation of bandwith capability of negative-feedback amplifiers (without simulation)
- Interpretation of simulation results
- Setting-up frequency compensation strategies for negative feedback amplifiers
- Understand effects of implementation of frequency compensation

Poles and zeros

Poles of a network

- Poles are the complex eigenfrequencies of a dynamic system
  - Solutions of characteristic equation of a system, does not require knowledge of input or output
  - Stable system: all poles have a negative real part
  - Number of poles in passive networks is sum of number of independent capacitor voltages and independent inductor currents
  - Poles found from MNA equation $\mathbf{I} = \mathbf{MV}$, by solving $\det \mathbf{M} = 0$
  - Alternatively, using $\mathbf{M} = \mathbf{G} + s\mathbf{C}$: $\det(\mathbf{G} + s\mathbf{C}) = 0$ or $\det(\mathbf{1} - s\mathbf{T}) = 0$, with $\mathbf{T} = \mathbf{G}^{-1}\mathbf{C}$:
    - If τ is an eigenvalue of $\mathbf{T}$, the complex frequency of the corresponding pole is $s = -\frac{1}{\tau}$

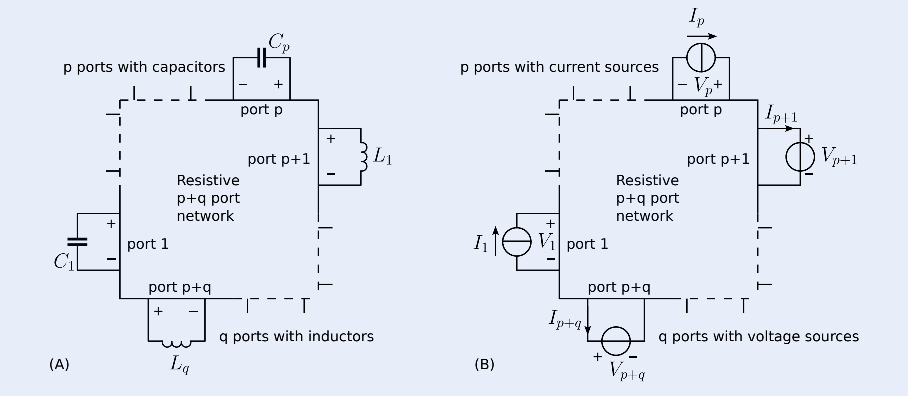
Poles

Estimation in networks without feedback or unity coupling factors

- Number of poles = number of capacitors plus number of inductors minus number of independent loops of capacitors and (controlled) voltage sources minus number of independent cut sets of inductors and (controlled) current sources
- Zero frequency poles (poles in the origin):
  - Number of independent cut sets of capacitors (and current sources) plus number of independent loops of inductors (and voltage sources)

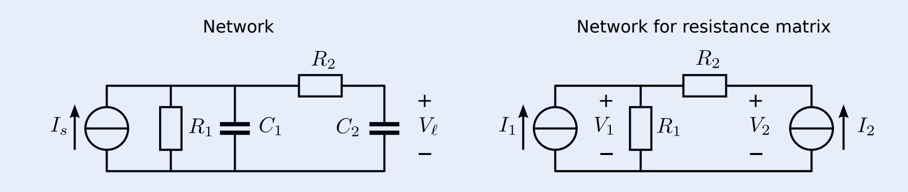
Poles

The resistance matrix



Poles

The resistance matrix, example



$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$\mathcal{R} = \begin{pmatrix} R_1 & R_1 \\ R_1 & R_1 + R_2 \end{pmatrix}$$

Poles

The time-constant matrix, example

- The time constant matrix is the product of the resistance matrix and the diagonal matrix $\mathcal{C}$:
$$\mathbf{T} = \mathcal{R}\mathcal{C}$$
- The diagonal matrix $\mathcal{C}$: $\mathcal{C}_{ii}$ is value of reactive element connected to port i. Negative value for inductor:

$$\mathcal{C} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$$

$$\mathbf{T} = \begin{pmatrix} C_1 R_1 & C_2 R_1 \\ C_1 R_1 & C_2 \left(R_1 + R_2\right) \end{pmatrix}$$

- Poles: $s = -\frac{1}{\tau}$: $\tau =$ eigenvalue of $\mathbf{T}$

Poles

The time-constant matrix, estimation

- Energy exchange between reactive elements makes is difficult to estimate poles
- If a capacitor at a port causes the dominant pole, that port can be considered shorted for frequencies above the frequency of the pole
- if an inductor at a port causes the dominant pole, that port can be considered open for frequencies above the frequency of the pole

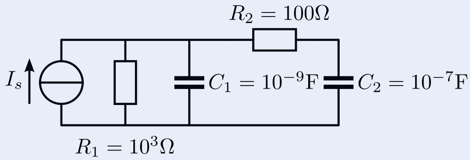
Poles

Estimation in networks: the time constant matrix based procedure

- Estimation of pole frequencies, with low charge exchange between capacitors and low flux exchange between inductors:
  1. Consider all time constants in the circuit, if there are no time constants: there are no poles
  2. If no resonance $(Q < 1)$ for the largest time constant, then largest time constant is that of dominant pole
  3. If dominant pole involves a capacitor: replace it with a short and goto (1)
  4. If dominant pole involves an inductor: replace it with an open circuit and goto (1)
  5. If resonance $(Q > 1)$ for the largest time constant, then we have two dominant complex conjugated poles (see book for frequency and quality factor), replace the capacitor of the resonator with a short and the inductor of the resonator with an open circuit and goto (1)

Poles

Numeric example



1. Time constants: $R_1 C_1 = 1\mu\text{s}$, $(R_1 + R_2)\, C_2 = 110\mu\text{s}$
2. Dominant pole caused by C_2: $p_1 = -9 \times 10^3 \text{rad/s}$
3. Short port with C_2
4. New time constant: $C_1 \frac{R_1 R_2}{R_1+R_2} = 90.9\text{ns}$: $p_2 = -11 \times 10^6 \text{rad/s}$
5. Eigenvalues of $\begin{pmatrix} 10^{-6} & 10^{-4} \\ 10^{-6} & 0.11 \times 10^{-3} \end{pmatrix}$: $0.111\mu\text{s}$ and 90.16ns
6. Small error due to energy exchange between capacitors

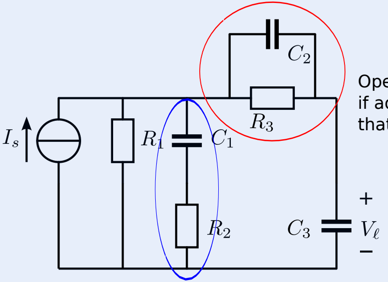
Zeros

Zeros of a transfer

- Zeros are the complex frequencies at which the transfer from input to output is zero:
  - A complex frequency at which there is a short in parallel with the signal path
  - A complex frequency at which there is an open circuit series with the signal path
  - A complex frequency at which input-output transfers via different paths cancel each other
  - Not always that simple to find!
- Zeros are found on poles that cannot be observed in an input-output transfer

Zeros

Example



Open circuit in series with signal path if admittance of capacitor opposite to that of resistor:

$$sC_2 = -\frac{1}{R_3}$$
$$s = -\frac{1}{R_3 C_2}$$

Short circuit in parallel with signal path if impedance of capacitor opposite to that of resistor:

$$R_2 = -\frac{1}{sC_1}$$
$$s = -\frac{1}{R_2 C_1}$$