Network Theory

Nodal Analysis and Modified Nodal Analysis

Anton J.M. Montagne

Persictec B.V.

March 6, 2017

Introduction

Why circuit analysis?

- Symbolic analysis of linear(ized) circuits is required for:
  1. Finding means to affect the performance the circuit
  2. Setting-up design equations for circuit element values
- Numerical analysis of circuits with given element values is required for:
  1. Numeric performance analysis
  2. Numeric performance optimization

Introduction

In which way do we perform circuit analysis

Symbolic analysis:SLiCAP

Limited to analysis of linear circuits:

1. Set-up network matrix equations using MNA
2. Derive and solve design equations for
   1. Noise
   2. DC offset and bias quantities
   3. Bandwidth
   4. Frequency compensation

Numerical analysis: SPICE

Analysis types:

- DC, OP: instantaneous fixed, nonlinear
- AC, NOISE, PZ: linear fixed dynamic
- TRAN: nonlinear timevariant dynamic
- Parametric analysis: combinations of the above.

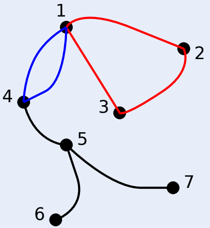
Introduction

Electrical networks

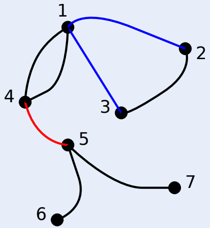
- An electric network consists of interconnected network elements
- Connections are called nodes
- Connecting elements between nodes form the branches
- A connected graph has at least one path among the branches that connects all the nodes
- A sub graph is a subset of branches with their corresponding nodes
- A closed path of branches is called a loop
- A collection of branches that isolates a sub graph wen removed is called a cut set
- A tree is a collection of branches that connects all the nodes but has no loops

Nodal analysis

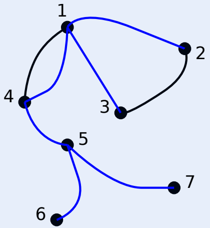
Graphs and nodal analysis



red: loop across 1,2,3
blue: loop across 1,4

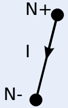


red: cut set isolating sub network 1,2,3,4
blue: cut set isolating sub network 2,3



blue: tree of the network

NODAL ANALYSIS



Sign convention: Positive current flows through branche from node with positive voltage to node with negative voltage

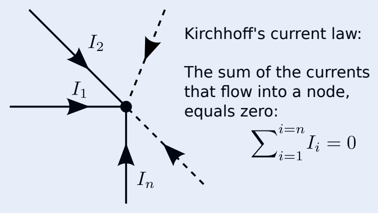
Two-terminal element relations in voltage controlled form:

$$I = Y(V_{N+} - V_{N-})$$

Sum of currents flowing from node equals zero: charge conservation

Nodal analysis

Setting-up the equations



$$\mathbf{I} = \mathbf{YV}$$

$\mathbf{I} =$ independent current vector

$\mathbf{Y} =$ admittance matrix

$\mathbf{V} =$ nodal voltage vector

1. Set-up circuit diagram
2. Number nodes $(0 \ldots n)$
3. 0 is reference node
4. Set up $(n-1)$ nodal equations (all nodes except ref. node)

- Network solution: $\mathbf{V} = \mathbf{Y}^{-1}\mathbf{I}$
- Transfer from an independent current I_k to a nodal voltage V_j

$$\frac{V_j}{I_k} = \left(\mathbf{Y}^{-1}\right)_{j,k} = \frac{\mathcal{C}_{kj}}{\det \mathbf{Y}}$$

Nodal analysis

Solving the equations

- The *cofactor* $\mathcal{C}_{kj}(\mathbf{M})$ of a matrix $\mathbf{M}$ is defined as the determinant of the minor matrix $\mathcal{M}_{k,j}$, multiplied with -1^{j+k}

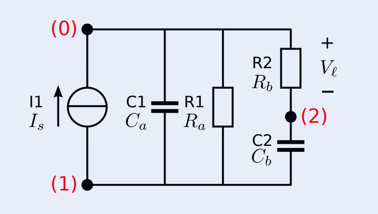
$$\mathcal{C}_{kj}(\mathbf{M}) = -1^{j+k}\det(\mathcal{M}_{k,j})$$

- The minor matrix $\mathcal{M}_{k,j}$ is the matrix $\mathbf{M}$ with the $k-th$ row and the $j-th$ column left out
- Poles and zeros of this transfer:

$$\begin{aligned} \text{poles} &: \det \mathbf{Y} = 0, \\ \text{zeros} &: \mathcal{C}_{kj}(\mathbf{Y}) = 0 \end{aligned}$$

Nodal analysis

Example



$$(1) \qquad 0 = I_s + V_1 sC_1 + V_1 \frac{1}{R_1} + (V_1 - V_2)\, sC_2$$

$$(2) \qquad 0 = V_2 \frac{1}{R_2} + (V_2 - V_1)\, sC_2$$

$$\begin{bmatrix} -I_s \\ 0 \end{bmatrix} = \begin{bmatrix} s\,(C_1 + C_2) + \frac{1}{R_1} & -sC_2 \\ -sC_2 & sC_2 + \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Nodal analysis

General form of the admittance matrix

The general form of a node equation for node k is:

$$\sum i_k = -\sum Y_{k,1} v_1 - \sum Y_{k,2} v_2 \quad \ldots + \sum Y_{k,k} v_k \quad \ldots - \sum Y_{k,n-1} v_{n-1}$$

in which:

$$\begin{aligned} \sum Y_{k,j} &= \text{sum of the admittances connected between node } k \text{ and node } j \\ \sum Y_{k,k} &= \text{sum of the admittances connected to node } k \\ \sum i_k &= \text{sum of the independent currents flowing into node } k. \end{aligned}$$

Nodal analysis

General form of the admittance matrix in words

Each diagonal element of the admittance matrix equals the sum of the admittances of each element connected to the corresponding node.

So, the first diagonal element is the sum of admittances connected to node 1, the second diagonal element is the sum of admittances connected to node 2, and so on.

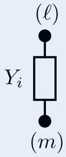
The off-diagonal elements are the sum of the negative admittances of the elements connected to the pair of corresponding nodes.

Hence, an admittance between nodes 1 and 2 appears in $\mathbf{Y}$ at the locations $(1,1)$ and $(2,2)$ with a positive sign, and at the locations $(1,2)$ and $(2,1)$ with a negative sign.

Nodal analysis

Admittance stamp

An admittance connected between nodes ℓ and m appears in $\mathbf{Y}$ at the locations (ℓ, ℓ) and (m, m) with a positive sign, and at the locations (ℓ, m) and (m, ℓ) with a negative sign.



$$
\mathrm{Y} = \begin{array}{c} \\ 1 \\ \ell \\ \\ m \\ n-1 \end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & \ell & m & n-1 \end{array} \\
\left[\begin{array}{cccccc}
\cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
\cdots & \cdots & Y_i & \cdots & -Y_i & \cdots \\
 & \vdots & \vdots & & \vdots & \vdots \\
\cdots & \cdots & -Y_i & \cdots & Y_i & \cdots \\
\cdots & \vdots & \cdots & \vdots & \cdots & \vdots
\end{array}\right]
\end{array}
$$

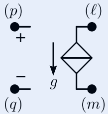
A network with passive elements only, has a symmetrical admittance matrix $Y_{jk} = Y_{kj}$.

Nodal analysis

Voltage-controlled elements only

Due to its structure, nodal analysis can be applied to networks having voltage-controlled elements only:

A voltage-controlled current source G_x, with its current flowing from node ℓ into node m, which is controlled by the voltage between node p (positive) and node q (negative), and with a gain of g [A/V], adds in the ℓ-th row: $+g$ in column p and $-g$ in column q, and in the m-th row: $-g$ in column p and $+g$ in column q.



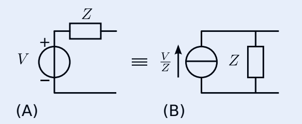
$$
\mathrm{Y}= \begin{array}{c} \\ 1 \\ \ell \\ m \\ n-1 \end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & p & q & n-1 \end{array} \\
\left[\begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\cdots & g & -g & \cdots \\
\cdots & -g & g & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{array}\right]
\end{array}
$$

Nodal analysis

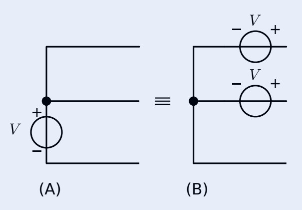
Use of network theorems

Network theorems can be applied to convert current-controlled elements such as voltage sources into voltage controlled elements.

Thévenin-Norton



Blakesley voltage-shift

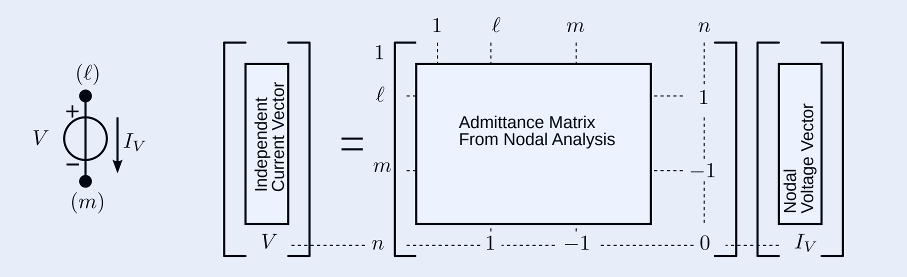


Modified Nodal Analysis

Introduction

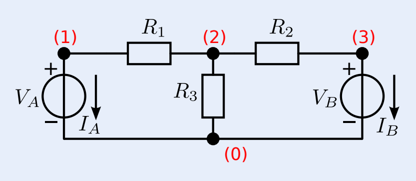
Current-controlled elements can be used within nodal analysis:

1. Add unknown branch currents to the vector with nodal voltages
2. Add equations: relate the nodal voltages to the branch voltage of the element



Modified Nodal Analysis

Example: setting-up equations



$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ V_A \\ V_B \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 0 & 1 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & 0 & 0 \\ 0 & -\frac{1}{R_2} & \frac{1}{R_2} & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ I_A \\ I_B \end{bmatrix}$$

Modified Nodal Analysis

Network solution

Equations

$$\begin{bmatrix} 0 & 0 & 0 & V_A & V_B \end{bmatrix}^T = \mathbf{M} \begin{bmatrix} V_1 & V_2 & V_3 & I_A & I_B \end{bmatrix}^T$$

Solution

$$\begin{bmatrix} V_1 & V_2 & V_3 & I_A & I_B \end{bmatrix}^T = \left(\mathbf{M}^{-1}\right) \begin{bmatrix} 0 & 0 & 0 & V_A & V_B \end{bmatrix}^T$$

$$\mathbf{M}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{R_1 R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} & 0 & \frac{R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} & \frac{R_1 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} \\ 0 & 0 & 0 & 0 & 1 \\ 1 & \frac{R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} & 0 & -\frac{R_2 + R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} & \frac{R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} \\ 0 & \frac{R_1 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} & 1 & \frac{R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} & -\frac{R_1 + R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} \end{bmatrix}$$

Modified Nodal Analysis

Network solution, simplified

Since there are no independent currents flowing into nodes 1, 2 and 3, the first three columns can be eliminated. The result then simplifies to:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ I_A \\ I_B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} & \frac{R_1 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} \\ 0 & 1 \\ -\frac{R_2 + R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} & \frac{R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} \\ \frac{R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} & -\frac{R_1 + R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} \end{bmatrix} \begin{bmatrix} V_A \\ V_B \end{bmatrix}$$

Modified Nodal Analysis

Cramer's rule

Solution of V_2 from previous solution:

$$V_2 = V_A \left(\mathbf{M}^{-1}\right)_{2,4} + V_B \left(\mathbf{M}^{-1}\right)_{2,5} = \frac{R_2 R_3 V_A + R_1 R_3 V_B}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

Alternatively: use Cramer's rule: If

$$\mathbf{X} = \mathbf{M}\mathbf{Y}$$

with $\mathbf{X}$ is a vector of independent variables and $\mathbf{Y}$ is a vector with dependent variables, then a dependent variable i.e. Y_i can be found as

$$Y_i = \frac{\det \mathbf{M}'}{\det \mathbf{M}}$$

In which $\mathbf{M}'$ is the matrix $\mathbf{M}$ in which the $i-th$ column has been replaced with $\mathbf{X}$.

Modified Nodal Analysis

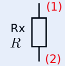
Cramer's rule, example

$$V_2 = \frac{\det \begin{bmatrix} \frac{1}{R_1} & 0 & 0 & 1 & 0 \\ -\frac{1}{R_1} & 0 & -\frac{1}{R_2} & 0 & 0 \\ 0 & 0 & \frac{1}{R_2} & 0 & 1 \\ 1 & V_A & 0 & 0 & 0 \\ 0 & V_B & 1 & 0 & 0 \end{bmatrix}}{\det \begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 0 & 1 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & 0 & 0 \\ 0 & -\frac{1}{R_2} & \frac{1}{R_2} & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}} = \frac{R_2R_3V_A + R_1R_3V_B}{R_1R_2 + R_1R_3 + R_2R_3}$$

Modified Nodal Analysis

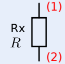
Stamps two-terminal

A



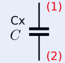
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{R} & -\frac{1}{R} \\ -\frac{1}{R} & \frac{1}{R} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

B



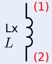
$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & R \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_{Rx} \end{bmatrix}$$

C



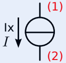
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} sC & -sC \\ -sC & sC \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

D



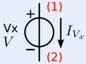
$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & sL \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_{L_x} \end{bmatrix}$$

E



$$\begin{bmatrix} I \\ -I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

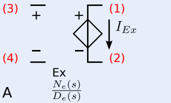
F



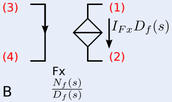
$$\begin{bmatrix} 0 \\ 0 \\ V \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_{Vx} \end{bmatrix}$$

Modified Nodal Analysis

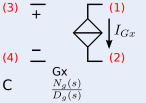
Stamps controlled sources



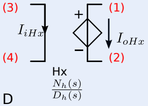
$$\begin{bmatrix} 0\\0\\0\\0\\0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ D_e(s) & -D_e(s) & -N_e(s) & N_e(s) & 0 \end{bmatrix} \begin{bmatrix} V_1\\V_2\\V_3\\V_4\\I_{Ex} \end{bmatrix}$$



$$\begin{bmatrix} 0\\0\\0\\0\\0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & D_f(s)\\ 0 & 0 & 0 & 0 & -D_f(s)\\ 0 & 0 & 0 & 0 & N_f(s)\\ 0 & 0 & 0 & 0 & -N_f(s)\\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1\\V_2\\V_3\\V_4\\I_{F_x} \end{bmatrix}$$



$$\begin{bmatrix} 0\\0\\0\\0\\0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & N_g(s) & -N_g(s) & -D_g(s) \end{bmatrix} \begin{bmatrix} V_1\\V_2\\V_3\\V_4\\I_{Gx} \end{bmatrix}$$

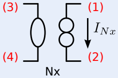


$$\begin{bmatrix} 0\\0\\0\\0\\0\\0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1- & 0\\ 0 & 0 & 1 & -1 & 0 & 0\\ D_h(s) & -D_h(s) & 0 & 0 & -N_h(s) & 0 \end{bmatrix} \begin{bmatrix} V_1\\V_2\\V_3\\V_4\\I_{iHx}\\I_{oHx} \end{bmatrix}$$

Modified Nodal Analysis

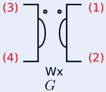
Stamps other

A



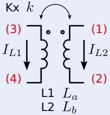
$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ I_{Nx} \end{bmatrix}$$

B



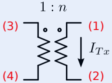
$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & G & -G \\ 0 & 0 & -G & G \\ -G & G & 0 & 0 \\ G & -G & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

C



$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -sL_a & -sk\sqrt{L_aL_b} \\ 0 & 0 & 1 & -1 & -sk\sqrt{L_aL_b} & -sL_b \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ I_{L_1} \\ I_{L_2} \end{bmatrix}$$

D

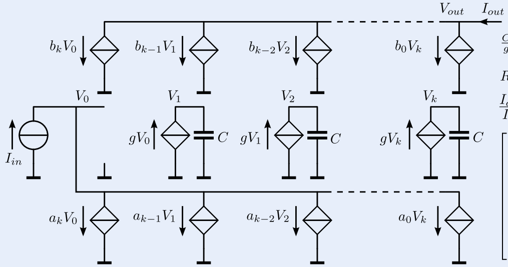


$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 & -n \\ 1 & -1 & n & -n & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ I_{Tx} \end{bmatrix}$$

Modified Nodal Analysis

Polynomial state equation implementation

- Implementation of polynomials using integrators
- Allows $\mathbf{M} = \mathbf{G} + s\mathbf{C}$ and $\mathbf{T} = \mathbf{G}^{-1}\mathbf{C}$
- Poles can be found from non-zero eigenvalues of $\mathbf{T}$
- DC solution required (non singular $\mathbf{G}$)



$$\frac{C}{g} = 1$$

$$R = 1$$

$$\frac{I_{out}}{I_{in}} = \frac{b_o + b_1 s + b_2 s^2 + \cdots + b_k s^k}{a_o + a_1 s + a_2 s^2 + \cdots + a_k s^k}$$

$$\begin{bmatrix} 0 \\ I_{in} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & b_k & b_{k-1} & b_{k-2} & \cdots & b_1 & b_0 \\ 0 & a_k & a_{k-1} & a_{k-2} & \cdots & a_1 & a_0 \\ 0 & -g & sC & 0 & \cdots & 0 & 0 \\ 0 & 0 & -g & sC & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & sC & 0 \\ 0 & 0 & 0 & 0 & \cdots & -g & sC \end{bmatrix} \begin{bmatrix} V_{out} \\ V_0 \\ V_1 \\ V_2 \\ \vdots \\ V_{k-1} \\ V_k \end{bmatrix}$$